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ORIGINAL ARTICLE

On (*m*, *n*)-absorbing prime ideals and (*m*, *n*)-absorbing ideals of commutative rings

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Abstract

Let R be a commutative ring with nonzero identity. In this paper, we introduce and investigate a generalization of 1-absorbing prime ideals. Let m, n be nonzero positive integers such that $m > n$. A proper ideal I of R is said to be an (m, n)-absorbing prime ideal if whenever nonunit elements $a_1, ..., a_m \in R$ and $a_1...a_m \in I$, then $a_1...a_n \in I$ or $a_{n+1}...a_m \in I$. We give some basic properties of this class of ideals and we study (m, n)-absorbing prime ideals of localization of rings, direct product of rings and trivial ring extensions. A proper ideal I of R is called an AB-(m, n)-absorbing ideal of R if whenever $a_1 \cdots a_m \in I$ for some elements $a_1, ..., a_m \in R$, then there are n of the a_i's whose product is in I. A proper ideal I of R is called an (m, n)-absorbing ideal of R if whenever $a_1 \cdots a_m \in I$ for some nonunit elements $a_1, ..., a_m \in R$, then there are n of the a_i's whose product is in I. We study some connections between (m, n)-absorbing prime ideals, (m, n)-absorbing ideals and AB-(m, n)-absorbing ideals of commutative rings.

Keywords 1-absorbing prime ideal · (m, n)-absorbing prime ideal · Prime ideal · Trivial ring extension

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1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. If R is a ring, then $\sqrt{I}$ denotes the *radical* of an ideal I of R and $J(R)$ denotes the Jacobson radical of R. A commutative ring R with exactly one maximal ideal is called a *quasi-local* ring.

The prime ideal, which is an important subject of ideal theory, has been widely studied by various authors. Among the many recent generalizations of the notion of prime ideals in the literature, we find the following, due to Badawi [5]. A proper ideal I of R is said to be a 2-*absorbing* ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In this case $\sqrt{I} = P$ is a prime ideal with $P^2 \subseteq I$ or $\sqrt{I} = P_1 \cap P_2$ where P_1, P_2 are incomparable prime ideals with $P_1P_2 \subseteq I$, cf. [5, Theorem 2.4]. A generalization of 2-absorbing ideals was studied in [1, 3, 4]. We recall from [3] that a proper ideal I of a commutative ring R is called an *n-absorbing* ideal of R for some positive integer $n \geq 1$ if whenever $a_1 \cdots a_{n+1} \in I$ for some elements $a_1, ..., a_{n+1} \in R$, then there are n of the a_i's whose product is in I. Let m, n be positive integers such that $m > n$ and I be a proper ideal of a commutative ring R. Then I is called an *(m, n)-closed* ideal of R as in [4] if whenever $x^m \in I$ for some $x \in R$, then $x^n \in I$. Furthermore, I is called an (m, n)-*absorbing* ideal of R as in [1] if whenever $a_1 \cdots a_m \in I$ for some nonunits $a_1, ..., a_m \in R$, then there are n of the a_i's whose product is in I. Recently, Badawi and Yetkin [6] consider a new class of ideals called the class of 1-absorbing primary ideals. A proper ideal I of a ring R is called a 1-*absorbing primary* ideal of R if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$. In [12], A. Yassine et. al introduced the concept of 1-absorbing prime ideals which is a generalization of prime ideals. A proper ideal I of R is a *1-absorbing prime* ideal if whenever nonunit elements $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $c \in I$. In this case $\sqrt{I} = P$ is a prime ideal, cf. [12, Theorem 2.3], and if R is a commutative ring that admits a 1-absorbing prime ideal that is not prime, then R is a quasi-local ring.

In this paper, we study a generalization of 1-absorbing prime ideals. Let m, n be nonzero positive integers such that $m > n$. A proper ideal I of a commutative ring R is said to be an (m, n)-*absorbing prime* ideal of R if whenever nonunit elements $a_1, ..., a_m \in R$ and $a_1 \cdots a_m \in I$, then $a_1 \cdots a_n \in I$ or $a_{n+1} \cdots a_m \in I$. Clearly, if I is an (m, n)-absorbing prime ideal, then I is an $(m+1, n+1)$-absorbing prime ideal. In particular, every prime ideal is an (m, n)-absorbing prime ideal. However, the converse is not true. Furthermore, we call I an *AB*-(m, n)-*absorbing* ideal of R if whenever $a_1 \cdots a_m \in I$ for some elements $a_1, ..., a_m \in R$, then there are n of the a_i's whose product is in I. It is clear that every *AB*-(m, n)-absorbing ideal of a commutative ring is an (m, n)-absorbing ideal as in [1]. We give an example (see Example 2.3) of a commutative ring R that admits an (m, n)-absorbing ideal that is not an *AB*-(m, n)-absorbing ideal. Let I be a proper ideal of a ring R such that $I \nsubseteq J(R)$. We show (Corollary 2.5) that I is an (m, n)-absorbing ideal of R if and only if I is an *AB*-(m, n)-absorbing ideal of R.

Among many results, we give an example of an (m, n)-absorbing prime ideal that is not an $(m-1, n-1)$-absorbing prime ideal for some nonzero positive integers $m > n$ (Example 2.9). We show (Theorem 2.10 (1)) that if a ring R is not quasi-local, then a proper ideal I of R is an (m, n)-absorbing prime ideal if and only if I is a prime ideal of R. If I is a proper ideal of R such that I is an (m, n)-absorbing prime ideal of R for some positive integers $m > n$, then we show (Theorem 2.13) that $\sqrt{I}$ is a prime ideal of R. If R is a quasi-local ring with maximal ideal M and $n \geq 2$ is a positive integer, then we provide a method (Theorem 2.14 and Theorem 2.17) on how to construct a $(2n, n)$-absorbing prime ideal of R that is not a prime ideal of R such that $\sqrt{I} \neq M$. Let R be a chained ring with maximal ideal M and $n \geq 2$. We show (Theorem 2.19) that if a proper ideal I of R is an $(n+1, n)$-absorbing prime ideal of R that is not a prime ideal of R, then $I = M^k$ for some positive integer k, $2 \leq k \leq n$. Finally, we study (m, n)-absorbing prime ideals of localization of rings, direct product of rings and trivial ring extensions.

2 Some properties and connections

We start this section by the following definitions.

Definition 2.1 Let m, n be nonzero positive integers such that $m > n$ and I be a proper ideal of a commutative ring R. Then

(1) I is called an (m, n)-*absorbing prime* ideal of R if whenever nonunit elements $a_1, ..., a_m \in R$ and $a_1 ... a_m \in I$, then $a_1 \cdots a_n \in I$ or $a_{n+1} \cdots a_m \in I$.
(2) I is called an (m, n)-*absorbing* ideal of R as in [1] if whenever $a_1 \cdots a_m \in I$ for some nonunits $a_1, ..., a_m \in R$, then there are n of the a_i's whose product is in I.
(3) I is called an *AB*-(m, n)-*absorbing* ideal of R if whenever $a_1 \cdots a_m \in I$ for some elements $a_1, ..., a_m \in R$, then there are n of the a_i's whose product is in I.

The proof of the following results follows from the definitions. Hence we omit the proof.

Theorem 2.2 *Let m, n be nonzero positive integers such that $m > n$ and I be a proper ideal of a commutative ring R. Then*

(1) *If I is an AB-(m, n)-absorbing ideal of R, then I is an (m, n)-absorbing ideal of R.*
(2) *Suppose that I is an (m, n)-absorbing prime ideal of R. Let $k = max\{n, m-n\}$. Then I is an (m, k)-absorbing ideal of R.*
(3) *If I is an AB-(m, n)-absorbing ideal of R, then I is an (m, n)-closed ideal of R.*

In the following example, we show that the converse of (1), (2) and (3) in Theorem 2.2 are not true.

Example 2.3 Let $A = \mathbb{Z}_2[[X, Y, Z]]$, $H = (X^2 + Y^2, Y^2 + Z^2)A$ and $R = A/H$. Consider the ideal $I = (H + (XYZ, X(Y + Z), Y(X + Z), X^2)A)/H$.

(1) We show that I is a (4, 2)-absorbing ideal of R that is not an AB-(4, 2)-absorbing ideal of R. Let x, y, z denote the elements $X + H, Y + H, Z + H$ of R, respectively. We show that I is not an AB-(4, 2)-absorbing ideal of R. Since $1 \cdot x \cdot y \cdot z \in I$ but neither $xy \in I$ nor $yz \in R$ nor $xz \in I$, we conclude that I is not an AB-(4, 2)-absorbing ideal of R. In order to show that I is a (4, 2)-absorbing ideal of R, we make the following observations. Let $S = \{x, y, z\}$ and $D = \{v \in R \mid v$ is a nonunit element of $R\}$. Then

(a) $x^2f = y^2f = z^2f \in I$ for every $f \in R$.
(b) Let a_1, a_2 be distinct elements of S. Then $a_1a_2 \notin I$. Since $x^2 = y^2 = z^2 \in I$ and $xyz \in I$, we have $a_1a_2v \in I$ for every $v \in D$.
(c) Since $x(y + z), y(x + z) \in I$, we have $z(x + y) = x(y + z) + y(x + z) = xy + xz + yx + yz = xz + yz \in I$.
(d) Since $x(y + z), y(x + z), z(x + y) \in I$ and $x^2 = y^2 = z^2 \in I$, we have $(x + y + z)v \in I$ for every $v \in D$.
(e) Let $a_1, a_2 \in S$ (a_1, a_2 need not be distinct). Then $(a_1 + a_2)^2 = 0 + H = H \in I$.
(f) Let $v \in D \setminus I$. Then $v = a + b$ for some $a, b \in D$ such that $ad \in I$ for every $d \in D$ and $bc \notin I$ for some $c \in D$. In view of (a)–(d), we conclude that b must be one of the following types.

  (i) Type I. $b = x$, or $b = y$, or $b = z$.
  (ii) Type II. $b = x + y$, or $b = x + z$, or $b = y + z$.

(g) Assume that $a_1a_2a_3a_4 \in I$ for some $a_1, a_2, a_3, a_4 \in D$. In view of (f), we consider the following three cases.

  (i) Case I. We may assume that a_1, a_2, a_3, a_4 are all of type I (as in (i)). Hence we may assume that $a_1 = x, a_2 = y, a_3 = z$. Then a_4 must equal to a_1 or a_2 or a_3. Since $x^2 = y^2 = z^2 \in I$, we may assume that $a_4 = a_1$. Thus $a_1a_4 = x^2 \in I$.
  (ii) Case II. We may assume that a_1, a_2, a_3, a_4 are all of type II (as in (ii)). Hence we may assume that $a_1 = x + y, a_2 = y + z, a_3 = x + z$. Then a_4 must equal to a_1 or a_2 or a_3. Since $a_1^2 = a_2^2 = a_3^2 = 0 + H \in I$ by (e), we may assume that $a_4 = a_1$. Thus $a_1a_4 = 0 + H \in I$.
  (iii) Case III. We may assume that some of the a_i's are of type I and some of the a_i's are of type II. In light of (a) and (e), we may assume that a_1, a_2, a_3, a_4 are distinct. Then by investigating all possibilities, we conclude that there are two of the a_i's whose product is in I by (c).

(2) Let $a_1 = x, a_2 = y, a_3 = x$, and $a_4 = z$. Then $a_1a_2a_3a_4 \in I$, but neither $a_1a_2 \in I$ nor $a_3a_4 \in I$. Hence I is not a (4, 2)-absorbing prime ideal of R.

(3) Since I is a (4, 2)-absorbing ideal of R, it is clear that I is a (4, 2)-closed ideal, but I is not an AB-(4, 2)-absorbing ideal of R.

Let R and I be as in Example 2.3. It is clear that R is a quasi-local ring and hence $I \subset J(R)$. We have the following result.

Theorem 2.4 *Let $m, n \geq 1$ be positive integers such that $m > n$, R be a commutative ring and I be a proper ideal of R such that $1 + i \notin U(R)$ for some $i \in I$. Then I is an AB-(m, n)-absorbing ideal of R if and only if I is an (m, n)-absorbing ideal of R.*

Proof If I is an AB-(m, n)-absorbing ideal of R, then it is clear that I is an (m, n)-absorbing ideal. Hence assume that I is an (m, n)-absorbing ideal of R. Suppose that $a_1 a_2 \cdots a_m \in I$ for some elements $a_1, a_2, ..., a_m \in R$. We may assume that for some $k \geq n + 2$, we have $a_k, a_{k+1}, ..., a_m$ are units of R and $a_1, a_2, ..., a_{k-1}$ are nonunit elements of R. Thus $a_1 a_2 \cdots a_{k-1} \in I$. Since $1 + i \notin U(R)$ for some $i \in I$, let $b_k = b_{k+1} = \cdots = b_m = i + 1$. Then $a_1 \cdots a_{k-1} b_k \cdots b_m = a_1 \cdots a_{k-1}(i + 1)^{(m-k+1)} \in I$. Since I is an (m, n)-absorbing ideal of R, there are n elements of the a_i's, $1 \leq i \leq k - 1$, and b_j's, $k \leq j \leq m - k + 1$, whose product is in I. Since $i \in I$ and $1 \notin I$, $(i + 1)^e \notin I$ for every positive integer $e \geq 1$. Hence there is an integer h, $1 \leq h \leq n$, and h elements of the a_i's, $1 \leq i \leq k - 1$, say $a_1, ..., a_h$, such that $a_1 \cdots a_h (i + 1)^{(n-h)} \in I$. Since $(i + 1)^{n-h} = fi + 1$ for some $f \in R$ (note that if $h = n$, then $f = 0$) and $a_1 \cdots a_h (fi + 1) \in I$, we conclude that $a_1 \cdots a_h \in I$. Thus I is an AB-(m, n)-absorbing ideal of R. □

In view of the proof of Theorem 2.4, we have the following corollary.

Corollary 2.5 *Let $m, n \geq 1$ be positive integers such that $m > n$, R be a commutative ring and I be a proper ideal of R such that $I \nsubseteq J(R)$. Then I is an AB-(m, n)-absorbing ideal of R if and only if I is an (m, n)-absorbing ideal of R.*

Proof Since $I \nsubseteq J(R)$, $1 + i \notin U(R)$ for some $i \in I$. Hence the claim is clear by Theorem 2.4. □

Let $m, n \geq 1$ be positive integers such that $m > n$. Assume that I_1 is an (m, n)-absorbing ideal of R_1 and I_2 is an (m, n)-absorbing ideal of R_2. Then $I_1 \times I_2$ needs not be an (m, n)-absorbing ideal of $R_1 \times R_2$. We have the following example.

Example 2.6 Let $R = \mathbb{Z} \times \mathbb{Z}$, $I_1 = 4\mathbb{Z}$ and $I_2 = 9\mathbb{Z}$. Then I_1 and I_2 are (4, 2)-absorbing ideals of $\mathbb{Z}$. We show that $I = I_1 \times I_2$ is not a (4, 2)-absorbing ideal of R. Let $x_1 = (2, 1), x_2 = (2, 1), x_3 = (1, 3), x_4 = (1, 3)$. Then $x_1, ..., x_4$ are nonunit elements of R and $x_1 \cdots x_4 \in I$, but the product of every two of the $x_i's$ is not in I. Thus I is not a (4, 2)-absorbing ideal of R.

In view of Example 2.6, we have the following result.

Theorem 2.7 *Let $m, n \geq 1$ be positive integers such that $m > n$, $R = R_1 \times R_2$, where R_1, R_2 are commutative rings with $1 \neq 0$, I_1 be a proper ideal of R_1, and $I = I_1 \times R_2$. The following statements are equivalent.*

(1) *I is an (m, n)-absorbing ideal of R.*
(2) *I_1 is an AB-(m, n)-absorbing ideal of R_1.*
(3) *I is an AB-(m, n)-absorbing ideal of R.*

Proof (1) ⇒ (2). Assume that I_1 is not an AB-(m, n)-absorbing ideal of R_1. Then there are $x_1, \cdots, x_m \in R_1$ such that $x_1 \cdots x_m \in I_1$, but there are no n of the $x_i's$ whose product is in I_1. Let $d_1 = (x_1, 0), d_2 = (x_2, 0), ..., d_m = (x_m, 0)$. Then $d_1, ..., d_m$ are non-unit elements of R and $d_1 \cdots d_m \in I$. Since I is an (m, n)-absorbing ideal of R, we conclude that there are n of the $d_i's$ whose product is in I. Hence there are n of the $x_i's$ whose product is in I_1, a contradiction. Thus I_1 is an AB-(m, n)-absorbing ideal of R_1.

(2) ⇒ (3). It is clear.

(3) ⇒ (4). Since every AB-(m, n)-absorbing ideal is an (m, n)-absorbing ideal, the claim is clear. □

The following remark follows from the definition of (m, n)-absorbing prime ideals.

Remark 2.8 Let R be a ring, I a proper ideal of R and let m, n be nonzero positive integers such that $m > n$.

(1) I is a (2, 1)-absorbing prime ideal of R if and only if I is a prime ideal.
(2) I is a (3, 2)-absorbing prime ideal of R if and only if I is a 1-absorbing prime ideal.
(3) If I is an (m, n)-absorbing prime ideal, then I is an $(m + 1, n + 1)$-absorbing prime ideal.
(4) I is an (m, n)-absorbing prime ideal if and only if I is an $(m, m - n)$-absorbing prime ideal.

The following example shows that the converse of Remark 2.8 (3) needs not be true.

Example 2.9 Let $R = \mathbb{Z}_{(p)}$, where (p) is a prime ideal of $\mathbb{Z}$ for a prime integer p of $\mathbb{Z}$, $n \geq 2$ be a nonzero positive integer, and $I = p^n R$. Assume that $a_1, ..., a_{n+1}$ are nonunit elements of R such that $a_1 \cdots a_{n+1} \in I$. Clearly, $a_i \in pR$ for each $i \in \{1, ..., n + 1\}$. Hence $a_1 \cdots a_n \in I$ and so I is an $(n + 1, n)$-absorbing prime ideal of R. Since $p^n \in I$ and $p^{n-1} \notin I$, we conclude that I is not an $(n, n - 1)$-absorbing prime ideal of R.

Let m, n be nonzero positive integers such that $m > n$. In the next result, we show that if a ring R admits an (m, n)-absorbing prime ideal that is not an $(m - 1, n - 1)$

-absorbing prime ideal of R, then R is a quasi-local ring. Furthermore, if R is not a quasi-local ring, then a proper ideal I of R is an (m, n)-absorbing prime ideal of R if and only if I is a prime ideal of R.

Theorem 2.10 *Let m, n be nonzero positive integers such that $m > n$, R be a ring and I a proper ideal of R. Then the following statements hold.*

(1) *Assume that R is not a quasi-local ring. Then the following conditions are equivalent.*

(a) *I is an (m, n)-absorbing prime ideal of R.*
(b) *I is an $(m-1, n-1)$-absorbing prime ideal of R*
(c) *I is an $(m-n+1, 1)$-absorbing prime ideal of R.*
(d) *I is a prime ideal of R.*

(2) *I is an $(n+1, n)$-absorbing prime ideal if and only if I is prime or R is quasi-local with maximal ideal M such that $M^n \subseteq I$.*

Proof (1) $(a) \Leftrightarrow (b)$. By Remark 2.8 (3), it suffices to prove that $(a) \Rightarrow (b)$. Assume that I is an (m, n)-absorbing prime ideal that is not an $(m-1, n-1)$-absorbing prime ideal of R. Hence there exist nonunit elements $a_1, ..., a_{m-1} \in R$ such that $a_1 \cdots a_{m-1} \in I$, $a_1 \cdots a_{n-1} \notin I$ and $a_n \cdots a_{m-1} \notin I$. Let d be a nonunit element of R. As $da_1 \cdots a_{m-1} \in I$, I is an (m, n)-absorbing prime ideal of R and $a_n \cdots a_{m-1} \notin I$, we conclude that $da_1 \cdots a_{n-1} \in I$. Let c be a unit element of R. Suppose that $d + c$ is a nonunit element of R. Since $(d + c)a_1 ..., a_{m-1} \in I$, I is an (m, n)-absorbing prime ideal of R and $a_n \cdots a_{m-1} \notin I$,, we get that $(d + c)a_1 \cdots a_{n-1} = da_1 \cdots a_{n-1} + ca_1 \cdots a_{n-1} \in I$. Since $da_1 \cdots a_{n-1} \in I$, we conclude that $a_1 \cdots a_{n-1} \in I$, which gives a contradiction. Hence, $d + c$ is a unit element of R. Now, the result follows from [6, Lemma 1].

$(b) \Leftrightarrow (c)$. This follows by induction.

$(c) \Leftrightarrow (d)$. Assume that I is $(m-n+1, 1)$-absorbing prime. Thus, by Remark 2.8 (4), I is an $(m-n+1, m-n)$-absorbing prime ideal and so "$(a) \Rightarrow (c)$" implies that I is prime. The converse is clear.

(2) It suffices to prove the " only if" assertion. If R is not quasi-local, then I is prime by (1). Now, assume that R is quasi-local with maximal ideal M such that $M^n \nsubseteq I$. We show that I is an $(n, n-1)$-absorbing prime ideal of R. Deny. Then there exist a nonunit elements $a_1, , ..., a_n$ of R such that $a_1 \cdots a_n \in I$, but neither $a_1 \cdots a_{n-1}$ nor $a_n \in I$. Let $x_1, ..., x_n \in M$. Since $x_1 \cdots x_n a_1 \cdots a_n \in I$, I is an $(n+1, n)$-absorbing prime ideal of R and $a_n \notin I$, we conclude that $x_1 \cdots x_n a_1 \cdots a_{n-1} \in I$. Also, as I is an $(n+1, n)$-absorbing prime ideal and $a_1 \cdots a_{n-1} \notin I$, we get that $x_1 \cdots x_n \in I$. Thus $M^n \subseteq I$, which is a contradiction. Therefore I is an $(n, n-1)$-absorbing prime ideal of R. Since $M^p \nsubseteq I$ for each $p \leq n$, we get, by induction, that I is a prime ideal of R. □

In view of Theorem 2.10 (1), we have the following result.

Corollary 2.11 *Let m, n be nonzero positive integers such that $m > n$. Suppose that I_1 and I_2 are ideals of the rings R_1 and R_2, respectively. Then the following statements are equivalent*:

(1) $I_1 \times I_2$ *is an* (m, n)*-absorbing prime ideal of* $R_1 \times R_2$.
(2) $I_1 \times I_2$ *is a prime ideal of* $R_1 \times R_2$.
(3) I_1 *is a prime ideal of* R_1 *and* $I_2 = R_2$ *or* $I_1 = R_1$ *and* I_2 *is a prime ideal of* R_2.

Proof Since $R_1 \times R_2$ is not quasi-local, the claim is clear by Theorem 2.10 (1). □

In view of Theorem 2.10 (2), we have the following result.

Corollary 2.12 *Let m, n be nonzero positive integers such that $m > n$. Assume that I is an $(n+1, n)$-absorbing prime ideal of a ring R that is not prime, and $I \subseteq J$ for some proper ideal J of R. Then J is an (m, n)-absorbing prime ideal of R.*

Proof Since I is an $(n+1, n)$-absorbing prime ideal of R that is not prime, we conclude that R is a quasi-local ring with maximal ideal M such that $M^n \subseteq I$. Since $M^n \subseteq I \subseteq J$, the claim is clear. □

Theorem 2.13 *Let m, n be nonzero positive integers such that $m > n$. If I is an (m, n)-absorbing prime ideal of a ring R, then $\sqrt{I}$ is a prime ideal of R. In particular, assume that I is not a prime ideal of R. Then R is a quasi-local ring with maximal ideal M and if $m = n + 1$, then $\sqrt{I} = M$.*

Proof Assume that I is an (m, n)-absorbing prime ideal of R. Let $x, y \in R$ such that $xy \in \sqrt{I}$. Without loss of generality, we may assume that x and y are nonunit elements of R. Thus, there exists a positive integer p such that $x^p y^p \in I$ and so $x^{n-1} x^p y^{m-n-1} y^p \in I$. Then, we can pick $a_i = x$ for $i = 1, ..., n-1$, $a_n = x^p$, $a_i = y$ for $i = n+1, ..., m-1$ and $a_m = y^p$. Since I is an (m, n)-absorbing prime ideal of R, we conclude that $x^{n+p-1} = a_1 \cdots a_n \in I$ or $y^{m-n+p-1} = a_{n+1} \cdots a_m \in I$. Therefore, $x \in \sqrt{I}$ or $y \in \sqrt{I}$. Hence $\sqrt{I}$ is a prime ideal of R. Assume that I is not a prime ideal of R. Then R is a quasi-local ring by Theorem 2.10 (1). Let M be the maximal ideal of R. If $m = n + 1$, then $M^n \in I$ by Theorem 2.10 (2). Thus $\sqrt{I} = M$. □

In view of Theorem 2.13, in the following result, we provide a method on how to construct (4, 2)-absorbing prime ideal of a quasi-local ring R with maximal ideal M such that I is not a prime ideal of R and $\sqrt{I} \neq M$.

Theorem 2.14 *Let R be a quasi-local ring with maximal ideal M. Assume that M contains a prime element x of R such that $M \neq xR$. Then $I = xM$ is a* (4, 2)*-absorbing prime ideal of R that is neither a* (3, 2)*-absorbing prime ideal of R nor a prime ideal of R such that $\sqrt{I} = xR \neq M$. Furthermore, I is an AB-*(4, 2)*-absorbing ideal of R that is an AB-*(3, 2)*-absorbing ideal of R.*

Proof It is clear that $\sqrt{I} \neq M$ and $\sqrt{I} = xR$ is a prime ideal of R. Since R is a quasi-local ring, $x \notin xM$, and hence xM is not a prime ideal of R. Assume that $a_1, ..., a_4 \in M$ such that $a_1 \cdots a_4 \in I$. Then at least one of the $a_i's$ is in xR. If $a_1 \in xR$ or $a_2 \in xR$, then $a_1a_2 \in I$. If $a_3 \in xR$ or $a_4 \in xR$, then $a_3a_4 \in I$. Thus I is a (4, 2)-absorbing prime ideal of R. We show that I is not a (3, 2)-absorbing prime ideal of R. Since $M \neq xR$, there is an $m \in M \setminus xR$. Thus $m^2 \notin xR$, and hence $m^2 \notin xM = I$. Let $a_1 = a_2 = m$ and $a_3 = x$. Then $a_1a_2a_3 \in I$, but neither $a_1a_2 \in I$ nor $a_3 \in I$. Hence I is not a (3, 2)-absorbing prime ideal of R.

We show that I is a (4, 2)-absorbing ideal of R. Assume that $a_1, ..., a_4 \in R$ such that $a_1 \cdots a_4 \in I$. Then at least one of the $a_i's$ is in xR. We may assume that $a_1 \in xR$. If a_2, a_3 and a_4 are unit elements of R, then $a_1 \in I$ and we are done. Assume $a_i \in M$ for some i, where $2 \leq i \leq 4$. Then $a_1a_i \in I$. Thus I is an AB-(4, 2)-absorbing ideal of R. By using a similar argument, one can show that I is an AB-(3, 2)-absorbing ideal of R. □

Remark 2.15 In view of [6, Theorem 6], Theorem 2.14, and Remark 2.8 (2), we conclude that $I = xM$, as in Theorem 2.14, is a 1-absorbing primary ideal of R that is neither a primary ideal of R nor a 1-absorbing prime ideal of R.

In view of Theorem 2.14, we have the following example.

Example 2.16 Let $A = \mathbb{Z}[X]$, and $L = (2, X)A$. Then L is a maximal ideal of A. Let $R = A_L$. Then R is a quasi-local ring with maximal ideal $M = (2, X)R$. Let $I = 2M = (4, 2X)R$. By Theorem 2.14, we conclude that I is a (4, 2)-absorbing prime ideal of R that is not a (3, 2)-absorbing prime ideal of R such that $\sqrt{I} = 2R \neq M$. Furthermore, I is an AB-(4, 2)-absorbing ideal of R that is an AB-(3, 2)-absorbing ideal of R.

In view of Theorem 2.14, we have the following result.

Theorem 2.17 *Let $n \geq 3$ be a positive integer and R be a quasi-local ring with maximal ideal M. Assume that M contains a prime element x of R such that $M \neq xR$. Then $I = xM^{n-1}$ is a $(2n, n)$-absorbing prime ideal of R that is not a prime ideal of R such that $\sqrt{I} = xR \neq M$. Furthermore, if $xM^{n-1} \neq xM^{n-2}$, then I is not a $(2n-1, n)$-absorbing prime ideal of R.*

Proof It is clear that $\sqrt{I} \neq M$ and $\sqrt{I} = xR$ is a prime ideal of R. Since R is a quasi-local ring, $x \notin xM$, and hence xM is not a prime ideal of R. Assume that $a_1, ..., a_n, ..., a_{2n} \in M$ such that $a_1 \cdots a_n \cdots a_{2n} \in I$. Then at least one of the $a_i's$ is in xR. If $a_i \in xR$ for some i, $1 \leq i \leq n$, then $a_1 \cdots a_n \in I$. If $a_i \in xR$ for some i, $n+1 \leq i \leq 2n$, then $a_{n+1} \cdots a_{2n} \in I$. Thus I is a ($2n$, n)-absorbing prime ideal of R. Assume that $xM^{n-1} \neq xM^{n-2}$. We show that I is not a $(2n-1, n)$-absorbing prime ideal of R. Since $M \neq xR$, there is an $m \in M \setminus xR$. Thus $m^n \notin xR$, and hence $m^n \notin xM^{n-1} = I$. Since $xM^{n-2} \neq xM^{n-1}$, there are $a_{n+2}, ..., a_{2n-1} \in M$ such that $xa_{n+2} \cdots a_{2n-1} \notin xM^{n-1}$. Let $a_1 = a_2 = \cdots = a_n = m$. Then $a_1 \cdots a_n x a_{n+2} \cdots a_{2n-1} \in I$, but neither $a_1 \cdots a_n = m^n \in I$ nor $xa_{n+2} \cdots a_{2n-1} \in xM^{n-1}$. Hence I is not a $(2n-1, n)$-absorbing prime ideal of R. □

Theorem 2.18 *Let I be an (m, n)-absorbing prime ideal of a ring R and let $d \in R \setminus I$ be a nonunit element of R. Then $(I : d) = \{x \in R \mid dx \in I\}$ is an $(m-1, n-1)$-absorbing prime ideal of R. In particular, for every proper ideal J of R with $J \nsubseteq I$, $(I : J)$ is an $(m-1, n-1)$-absorbing prime ideal of R.*

Proof Suppose that $a_1 \cdots a_{m-1} \in (I : d)$ for some nonunit elements $a_1, ..., a_{m-1}$ of R. Assume that $a_1 \cdots a_{n-1} \notin (I : d)$. Since $da_1 \cdots a_{m-1} \in I$ and I is an (m, n)-absorbing prime ideal of R, we conclude that $a_n \cdots a_{m-1} \in I \subseteq (I : d)$ and this completes the proof. □

Recall that a ring R is a chained ring if the set of all ideals of R are linearly ordered by inclusion. Moreover, R is said to be an arithmetical ring if R_M is a chained ring for each maximal ideal M of R. We next determine the $(n+1, n)$-absorbing prime ideals of a chained ring.

Theorem 2.19 *Let R be a chained ring with maximal ideal M and I be an $(n+1, n)$-absorbing prime ideal of R for some positive integer $n \geq 2$. If I is not a prime ideal of R, then $I = M^k$ for some positive integer k, $2 \leq k \leq n$. Furthermore, if $k < n$, then I is an $(k+1, k)$-absorbing prime ideal of R.*

Proof Let I be an $(n+1, n)$-absorbing prime ideal of R that is not a prime ideal of R. Since R is a quasi-local ring with maximal ideal M, we conclude that $M^n \subseteq I$ by Theorem 2.10 (2). Let $k = min\{i | M^i \subseteq I\}$. We show that $I = M^k$. Suppose that $M^k \subsetneq I$. Thus there is $a \in M^{k-1} \setminus I$ and $b \in I \setminus M^k$. Since R is a chained ring, we conclude that $b \in aR$. Hence $b = ar$ for some $r \in M$. Thus $b \in M^k$, a contradiction. Hence $I = M^k$. It is clear that M^k is a $(k+1, k)$-absorbing prime ideal of R. □

In view of Theorem 2.19, we have the following result.

Corollary 2.20 *Let R be an arithmetical ring with Jacobson radical M and I be a $(n+1, n)$-absorbing prime ideal of R for some positive integer $n \geq 2$. If I is not a prime ideal of R, then $I = M^k$ for some positive integer k, $2 \leq k \leq n$. Furthermore, if $k < n$, then I is a $(k+1, k)$-absorbing prime ideal of R.*

Proof If R is not a quasi-local ring, then I is a prime ideal of R by Theorem 2.10 (1). Thus assume that R is a quasi-local ring. Then M is the maximal ideal of R. Since R is an arithmetical quasi-local ring, we conclude that R is a chained ring. Thus $I = M^k$ for some positive integer k, $2 \leq k \leq n$ by Theorem 2.19. □

Theorem 2.21 *Let R be a ring, and I a proper ideal of R. Then I is an (m, n)-absorbing prime ideal if and only if whenever $I_1 \cdots I_m \subseteq I$ for some proper ideals $I_1, ..., I_m$ of R, then $I_1 \cdots I_n \subseteq I$ or $I_{n+1} \cdots I_m \subseteq I$.*

Proof It suffices to prove the "if" assertion. Suppose that I is an (m, n)-absorbing prime ideal and let $I_1, ..., I_m$ be proper ideals of R such that $I_1 \cdots I_m \subseteq I$ and $I_{n+1} \cdots I_m \nsubseteq I$. Hence $a_{n+1} \cdots a_m \notin I$ for some $a_{n+1} \in I_{n+1}, ..., a_m \in I_m$. Let $b_1 \in I_1, ..., b_n \in I_n$. Then $b_1 \cdots b_n a_{n+1} \cdots a_m \in I$. Since I is an (m, n)-absorbing ideal of R and $a_{n+1} \cdots a_m \notin I$, we conclude that $b_1 \cdots b_n \in I$. Thus $I_1 \cdots I_n \subseteq I$. □

Theorem 2.22 *Let R be a ring with Jacobson radical M and let n be a nonzero positive integer. The following statements are equivalent.*

(1) *Every proper ideal of R is an* $(n+1, n)$*-absorbing prime ideal.*
(2) *Every proper principal ideal of R is an* $(n+1, n)$*-absorbing prime ideal.*
(3) *R is quasi-local and* $M^n = 0$.

Proof (1) ⇒ (2). This is obvious.

(2) ⇒ (3). Assume that R is not a quasi-local ring. Then every proper principal ideal of R is a prime ideal by Theorem 2.10 (1). Consequently, R is a field, a contradiction. This implies that R is a quasi-local ring with maximal ideal M. Hence $\{0\}$ is a prime ideal or $M^n = \{0\}$ by Theorem 2.10 (2). Assume that $M^n \neq \{0\}$. Then R is an integral domain and there is some nonzero $x \in M^n$. It follows from Theorem 2.10 (2) that x^2R is a prime ideal or $M^n \subseteq x^2R$. If x^2R is a prime ideal, then $x^2R = xR$. If $M^n \subseteq x^2R$, then $M^n \subseteq x^2R \subseteq xR \subseteq M^n$, and thus $x^2R = xR$. Hence in both cases, we have $x^2R = xR$, and thus x is a unit, a contradiction.

(3) ⇒ (1). Let I be an ideal of R and $a_1, ..., a_{n+1}$ be nonunit elements of R such that $a_1 \cdots a_{n+1} \in I$. Since $M^n = 0$, we get that $a_1 \cdots a_n = 0 \in I$. Therefore, I is an $(n+1, n)$-absorbing prime ideal of R. □

Theorem 2.23 *Let* $f : R \to S$ *be a ring homomorphism. Suppose that f(a) is nonunit in S for every nonunit element a in R. Then the following statements hold.*

(1) *If J is an* (m, n)*-absorbing prime ideal of S, then* $f^{-1}(J)$ *is an* (m, n)*-absorbing prime ideal of R.*
(2) *If f is an epimorphism and I is a proper ideal of R containing* $\ker(f)$, *then I is an* (m, n)*-absorbing prime ideal of R if and only if f(I) is an* (m, n)*-absorbing prime ideal of S.*

Proof (1) Assume that $a_1 \cdots a_m \in f^{-1}(J)$, for some nonunit elements $a_1, ..., a_m \in R$. Then $f(a_1) \cdots f(a_m) \in J$. Thus $f(a_1) \cdots f(a_n) \in J$ or $f(a_{n+1}) \cdots f(a_m) \in J$, which implies that $a_1 \cdots a_n \in f^{-1}(J)$ or $a_{n+1} \cdots a_m \in f^{-1}(J)$. Therefore, $f^{-1}(J)$ is an (m, n)-absorbing prime ideal of R.

(2) Suppose that $f(I)$ is an (m, n)-absorbing prime ideal of S. Since $I = f^{-1}(f(I))$, we conclude that I is an (m, n)-absorbing prime ideal of R by (1). Conversely, let $x_1 \cdots x_m$ be nonunit elements of S with $x_1 \cdots x_m \in f(I)$. Then there exist $a_1, ..., a_m \in R$ such that $x_1 = f(a_1), ..., x_m = f(a_m)$ with $f(a_1 \cdots a_m) = x_1 \cdots x_m \in f(I)$. Since

$\ker(f) \subseteq I$, we have $a_1 \cdots a_m \in I$. Since I is an (m, n)-absorbing prime ideal of R and $a_1 \cdots a_m \in I$, we conclude that $a_1 \cdots a_n \in I$ or $a_{n+1} \cdots a_m \in I$, and thus $x_1 \cdots x_n \in f(I)$ or $x_{n+1} \cdots x_m \in f(I)$. Hence $f(I)$ is an (m, n)-absorbing prime ideal of S. □

In view of Theorem 2.23, we have the following result.

Corollary 2.24 *Let R be a ring, and $I \subseteq J$ be proper ideals of R. Assume that $a + I$ is a nonunit element of $\frac{R}{I}$ for every nonunit element $a \in R$. Then J is an (m, n)-absorbing prime ideal of R if and only if $\frac{J}{I}$ is an (m, n)-absorbing prime ideal of $\frac{R}{I}$.*

Theorem 2.25 *Let S be a multiplicatively closed subset of a ring R. If I is an (m, n)-absorbing prime ideal of R such that $I \cap S = \emptyset$, then I_S is an $(m-1, n-1)$-absorbing prime ideal of R_S. In particular, if I is a 1-absorbing prime ideal of R, then I_S is a 1-absorbing prime ideal of R_S.*

Proof Let I be an (m, n)-absorbing prime ideal of R such that $I \cap S = \emptyset$ and $\frac{a_1}{s_1} \cdots \frac{a_{m-1}}{s_{m-1}} \in I_S$ for some nonunit elements $a_1, ..., a_{m-1} \in R$ and $s_1, ..., s_{m-1} \in S$ such that $\frac{a_1}{s_1} \cdots \frac{a_{n-1}}{s_{n-1}} \notin I_S$. Then $ta_1 \cdots a_{m-1} \in I$ for some $t \in S$. Since I is (m, n)-absorbing prime and $ta_1 \cdots a_{n-1} \notin I$, we conclude that $a_n \cdots a_{m-1} \in I$. Thus $\frac{a_n}{s_n} \cdots \frac{a_{m-1}}{s_{m-1}} \in I_S$, which completes the proof. □

Let S be a multiplicatively closed subset of a ring R and I an ideal of R. The next example shows that if I_S is an (m, n)-absorbing prime ideal of R_S, then I needs not be an (m, n)-absorbing prime ideal of R.

Example 2.26 Let $p \neq q$ be two prime numbers. Set $I = p^2\mathbb{Z}$. Since $pqpq \in I$ and $pq \notin I$, I is not a (4, 2)-absorbing prime ideal of $\mathbb{Z}$. Now, let $S = \mathbb{Z} \setminus p\mathbb{Z}$ and $R = \mathbb{Z}_S$. Assume that $a_1, ..., a_4 \in pR$ and $a_1 \cdots a_4 \in I_S = p^2R$. Then it is clear that $a_1a_2 \in I_S$. Hence $I_S = p^2R$ is a (4, 2)-absorbing prime ideal of R.

Let A be a ring and E be an A-module. Then $A \ltimes E$, is called the *trivial* (*ring*) *extension of A by E*. We recall that $A \ltimes E$ is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a, e)(b, f) = (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known as the *idealization* $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [9, 10]. Trivial ring extensions have been studied and generalized by many authors (for example, cf. [2, 7, 8, 11]). We recall that if I is an ideal of A and F is a submodule of E, then $I \ltimes F$ is an ideal of $A \ltimes E$ if and only if $IE \subseteq F$. In then next result, we study (m, n)-absorbing prime ideals of trivial ring extensions.

Theorem 2.27 *Let A be a ring, E be an A-module, I be an ideal of A and F be a submodule of E such that $IE \subseteq F$. Then the following statements hold.*

(1) *If $I \ltimes F$ is an (m, n)-absorbing prime ideal of $A \ltimes E$, then I is an (m, n)-absorbing prime ideal of A.*
(2) *$I \ltimes E$ is an (m, n)-absorbing prime ideal of $A \ltimes E$ if and only if I is an (m, n)-absorbing prime ideal of A.*
(3) *$I \ltimes F$ is an $(n+1, n)$-absorbing prime ideal of $A \ltimes E$ if and only if one of the following conditions holds*:

(a) *I is prime and $F = E$.*
(b) *A is a quasi-local ring with maximal ideal M such that $M^n \subseteq I$ and $M^{n-1}E \subseteq F$ for $n \geq 2$.*

Proof (1) Assume that $I \ltimes F$ is an (m, n)-absorbing prime ideal of $A \ltimes E$ and let $a_1, ..., a_m$ be nonunit elements of A such that $a_1 \cdots a_m \in I$. Thus $(a_1, 0) \cdots (a_m, 0) = (a_1 \cdots a_m, 0) \in I \ltimes F$ which implies that $(a_1, 0) \cdots (a_n, 0) \in I \ltimes F$ or $(a_{n+1}, 0) \cdots (a_m, 0) \in I \ltimes F$. Therefore $a_1 \cdots a_n \in I$ or $a_{n+1} \cdots a_m \in I$ and so (1) holds.

(2) By (1), it suffices to prove the "if" assertion. Let $(a_1, e_1), ...(a_m, e_m)$ be nonunit elements of $A \ltimes E$ such that $(a_1, e_1) \cdots (a_m, e_m) \in I \ltimes E$. Clearly, $a_1 \cdots a_m \in I$ and so $a_1 \cdots a_n \in I$ or $a_{n+1} \cdots a_m \in I$ since I is an (m, n)-absorbing prime ideal of A. As $(a_1, e_1) \cdots (a_n, e_n) = (a_1 \cdots a_n, c)$ for some $c \in E$, we conclude that $(a_1, e_1) \cdots (a_n, e_n) \in I \ltimes E$. A similar argument shows that $(a_{n+1}, e_{n+1}) \cdots (a_m, e_m) \in I \ltimes E$. Therefore, $I \ltimes E$ is an (m, n)-absorbing prime ideal of $A \ltimes E$.

(3) Set $R = A \ltimes E$ and assume that $I \ltimes F$ is an $(n+1, n)$-absorbing prime ideal of R. So, Theorem 2.10 (2) implies that $I \ltimes F$ is a prime ideal of R or R is quasi-local with maximal ideal N such that $N^n \subseteq I \ltimes F$. By [2, Theorem 3.2 (2)], if $I \ltimes F$ is prime, then I is a prime ideal of A and $E = F$. In the remaining case, [2, Theorem 3.2 (1)] implies that A is quasi-local with maximal ideal M such that $N = M \ltimes E$. Let $a_1, ..., a_n \in M$. As $(a_1, 0) \cdots (a_n, 0) \in N^n$ and $N^n \subseteq I \ltimes F$, we get that $a_1 \cdots a_n \in I$ and thus $M^n \subseteq I$. Now, let $a_1, ..., a_{n-1} \in M$ and $e \in E$. Since $(a_1, 0) \cdots (a_{n-1}, 0)(0, e) = (0, a_1 \cdots a_{n-1}e) \in N^n \subseteq I \ltimes F$, we conclude that $a_1 \cdots a_{n-1}e \in F$ and thus $M^{n-1}E \subseteq F$. The converse follows by a similar reasoning. □

The next corollary is an immediate application of part (3) of Theorem 2.27.

Corollary 2.28 *Let A be a ring, E be an A-module, I be an ideal of A and F be a submodule of E such that $IE \subseteq F$. Then $I \ltimes F$ is a 1-absorbing prime ideal of $A \ltimes E$ if and only if I a prime ideal of A and $E = F$ or A is quasi-local with maximal ideal M such that $M^2 \subseteq I$ and $ME \subseteq F$.*

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Declarations

Conflict of interest The authors declare no conflict of interest.

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